ASYMPTOTIC CONNECTIVITY OF LOW DUTY-CYCLED WIRELESS SENSOR NETWORKS

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ABSTRACT

In this paper we study the asymptotic connectivity of a low duty-cycled wireless sensor network, where all sensors are randomly duty-cycled such that they are on/active at any time with a fixed probability. A wireless network is often said to be asymptotically connected if there exists a path from every node to every other node in the network with high probability as the network density approaches infinity. Within the context of a low duty-cycled wireless sensor network, the network is said to be asymptotically connected if for all realizations of the random duty-cycling (i.e., the combination of on and off nodes) there exists a path of active nodes from every node to every other node in the network with high probability as the network density approaches infinity. With this definition, we derive conditions under which a low duty-cycled sensor network is asymptotically connected. These conditions essentially specify how the nodes' communication range and the duty-cycling probability should scale as the network grows in order to maintain connectivity.

I. INTRODUCTION

The Army's Future Combat Systems potentially rely heavily on the efficient use of unattended sensors to detect, identify and track targets in order to enhance situation awareness, agility and survivability. Among different types of sensors, the unattended ground sensors (UGS) are typically deployed and left to self-organize and carry out various sensing, monitoring, surveillance and communication tasks. These sensors are operated on battery power, and energy is not always renewable due to cost, environmental and form-size concerns. This imposes a stringent energy constraint on the design of the communication architecture, communication protocols, and the deployment and operation of these sensors. It is thus critical to operate these sensors in a highly energy efficient manner.

It has been observed that low power sensors consume significant amount of energy while idling in addition to that consumed during transmission and reception. Consequently, it has been widely considered a key method of energy conservation to turn off sensors that are not actively involved in sensing or communication. By functioning at a *low duty cycle*, i.e., by reducing the fraction of time that a sensor is active/on, sensors are able to conserve energy, which consequently leads to prolonged lifetime. This is particularly applicable in scenarios where sensors are naturally idle for most of the time (e.g., detection of infrequent events such as fire, fault, etc., and transmission of very short messages). However, as sensors alternate between sleep and wake modes, its coverage and communication capability are inevitably disrupted. Duty-cycling sensory devices directly leads to loss of sensing coverage, while duty-cycling radio transceivers directly leads to loss of network connectivity. It is therefore crucial to understand the performance degradation as a result of duty-cycling the sensor nodes, and to design good networking mechanisms that work well with low duty-cycled sensor networks.

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In this paper we aim at understanding the fundamental relationship between duty-cycling the radio transceivers and the resulting network connectivity. Specifically we will consider *random* duty-cycling where sensor nodes are on/awake with a certain probability (called the wake/active probability). The definition of connectivity refers to the existence of a route (consisting of active nodes) from each *active* node to every other *active* node in the network. While intuitively increasing nodes' transmission radius and decreasing nodes' active probability have opposite effects on the connectivity, it is less clear how they are related quantitatively to ensure connectivity. We will focus on understanding how these quantities scale as the network density increases, by studying the asymptotic connectivity of the network. Asymptotic connectivity in this context refers to the existence of a route (consisting of active nodes) from each *active* node to every other *active* node in the network, as the number of nodes approaches infinity.

More precisely, we consider the network with n nodes uniformly and independently placed in a unit square in $\Re^2$. Each node is awake with probability $p(n)$ and is connected to active neighbors within the range of transmission $R(n)$ when it is active. The problem under consideration is how $p(n)$ and $R(n)$ are related to ensure that the network is connected with high probability as n goes to infinity. An important prior work is [1]. Our network model is essentially the same as that studied in [1], with the only difference that in [1] the wake/active probability $p(n)$ is always 1. [1] showed that it is sufficient and necessary for each node to be connected to $\Theta(\log n)$ nearest neighbors to achieve asymptotic connectivity as n approaches infinity. Building on this result, in this study we show that the above randomly duty-cycled network is asymptotically connected with probability one if and only if the *average* number of active neighbors a node has is on the order of $\log(np(n))$. It has to be mentioned that this result cannot be obtained as a straightforward extension to [1] as discussed in more detail in subsequent sections.

The rest of the paper is organized as follows. We present the network model and our main result in the next section, along with a discussion on its relationship to the related work. In Section III we give a number of preliminary results, and Section IV outlines the proof of the main result. Section V concludes the paper. Due to space limit, most of the proofs are not included, but they can be found in [2].

II. NETWORK MODEL, MAIN RESULT AND DISCUSSION

Consider a unit square in $\Re^2$, where n nodes are deployed uniformly and independently. Time is slotted. In each time slot, a node has a probability $p(n)$ of being awake or active, referred to as the *active probability*. An active node is connected to its active neighbors within a circle of radius $R(n)$, referred to as the transmission range. Such a network is said to be asymptotically connected if there exists a path of active nodes between any pair of two active nodes with high probability as the density n approaches infinity. In order to study the conditions under which such a network is asymptotically connected, we will utilize a number of results derived for a similar, but not duty-cycled network (i.e., where $p(n) = 1$ for all n). We begin by introducing the following types of networks/graphs that will be used in this paper.

- $\mathcal{G}_p(n, R(n))$ denotes the duty-cycled network mentioned above, i.e., a network formed in a unit square where n nodes are deployed uniformly and independently. In this network a node is active with probability $p(n)$ and when active is connected to its active neighbors within a circle of radius $R(n)$.
- $\mathcal{G}(n, R(n))$ denotes a non-duty-cycled network formed in a unit square with n nodes deployed uniformly and independently. In this network a node is always active and is connected to neighbors within a circle of radius $R(n)$.
- $\mathcal{G}^\lambda(n, R(n))$ denotes a network formed as a Poisson point process with intensity n. In this network a node is always active and is connected to neighbors within a circle of radius $R(n)$.
- $\mathcal{F}(n, \phi_n)$ denotes a network formed in a unit square with n nodes deployed uniformly and independently. In this network a node is always active and is connected to its ϕ_n nearest neighbors.
- $\mathcal{F}^\lambda(n, \phi_n)$ denotes a network formed as a Poisson point process with intensity n. In this network a node is always active and is connected to its ϕ_n nearest neighbors.

The following notations are used throughout this paper. For two functions $f(n)$ and $g(n)$ defined on some subset of the real line, (1) $f(n) = O(g(n))$ implies that there exist numbers n_0 and M such that $|f(n)| \leq M \cdot |g(n)|$ for all $n > n_0$ (asymptotic upper bound); (2) $f(n) = \Theta(g(n))$ implies that $f(n) = O(g(n))$ and $g(n) = O(f(n))$ (asymptotic tight bound); and (3) $f(n) = o(g(n))$ implies that $\lim_{n\to\infty} f(n)/g(n) = 0$ (asymptotically negligible).

Our main result is shown in the following theorem.

Theorem 1 *There exist two constants k_1 and k_2, $0 < k_1 < k_2$, such that:*

1) *for $np(n)R^2(n) = k_2 \log(np(n))$, we have*

$$\lim_{n\to\infty} Pr\{\mathcal{G}_p(n, R(n)) \text{ is connected } \} = 1 \; , \qquad (1)$$

2) *for $np(n)R^2(n) = k_1 \log(np(n))$, we have*

$$\lim_{n\to\infty} Pr\{\mathcal{G}_p(n, R(n)) \text{ is disconnected } \} = 1 \; . \qquad (2)$$

Eqn. (1) is also commonly viewed as a *sufficient* condition on connectivity and Eqn. (2) commonly viewed as a *necessary* condition on connectivity. Put together, $np(n)R^2(n) = \Theta(\log(np(n))$ can be viewed as the sufficient and necessary conditions for asymptotic connectivity. In subsequent sections we will also refer to these two equations as part I and part II of the theorem.

Below we sketch the idea of the proof of the above theorem and discuss this result within the context of other existing results on asymptotic connectivity.

Figure 1 summarizes the main idea of the proof, and illustrates where our technical contributions lie. The network we are interested in, $\mathcal{G}_p(n, R(n))$, is shown on the top left. To prove the theorem, we first show that if a Poisson network with intensity $np(n)$, i.e., $\mathcal{G}^\lambda(np(n), R(n))$, is asymptotically connected/disconnected given the condition $np(n)R(n)^2 = k\log(np(n))$ for some $k > 0$,

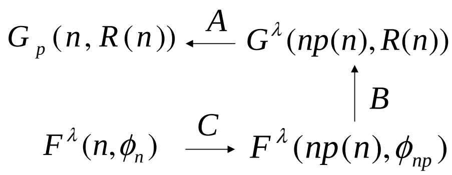

Fig. 1. Outline of the proof of Theorem 1.

then $\mathcal{G}_p(n, R(n))$ is asymptotically connected/disconnected given the same condition (for possibly different constants).

This process is illustrated by the arrow labeled with "A" in the figure. Conceptually, because of the random duty-cycling, there are only on average $np(n)$ nodes awake in the network at any instance of time. This makes the network $\mathcal{G}_p(n, R(n))$ behave like a Poisson network rather than one with a fixed number of nodes. However, in order to study asymptotic connectivity $np(n)$ needs to approach infinity, which renders inapplicable the standard result of approximating a binomial distribution with a Poisson distribution (which assumes a finite intensity). Although this seems a highly intuitive result, we were not able to find a prior proof. We give one such proof in Lemma 3, where we establish the Poisson approximation of a binomial distribution when $np(n) \to \infty$.

We next show that if the network $\mathcal{F}^\lambda(np(n), \phi_{np})$, i.e., a Poisson network with intensity $np(n)$ where each node is connected to its ϕ_{np} nearest neighbors, is asymptotically connected/disconnected given the condition $\phi_{np} = c\log(np(n))$, for some $c > 0$, then the network $\mathcal{G}^\lambda(np(n), R(n))$ is asymptotically connected/disconnected given the condition $np(n)R(n)^2 = k\log(np(n))$ for some $k > 0$.

This process is illustrated by the arrow labeled with "B" in the figure. Here $\mathcal{F}^\lambda(np(n), \phi_{np})$ is a Poisson network with ϕ_{np} neighbors for each node, and $\mathcal{G}^\lambda(np(n), R(n))$ is a Poisson network with neighbors within a finite radius $R(n)$ of each node. Note that for the latter, the condition $np(n)R(n)^2 = k\log(np(n))$ for some $k > 0$ is on the average number of neighbors a node has, whereas for the former the condition $\phi_{np} = c\log(np(n))$ for some $c > 0$ is on the actual number of neighbors a node has.

The last step is to show that network $\mathcal{F}^\lambda(np(n), \phi_{np})$ is asymptotically connected/disconnected given the condition $\phi_{np} = c\log(np(n))$, for some $c > 0$. This network is essentially the same as $\mathcal{F}^\lambda(n, \phi_n)$ (with a different intensity). This result is obtained in similar ways as in [3], which showed the same result for $\mathcal{F}(n, \phi_n)$. This step is illustrated by the arrow labeled with "C" in the figure.

Two most relevant results to that studied in this paper are from [1] and [3], respectively. In particular, as mentioned above [1] studied a network of the type $\mathcal{F}(n, \phi_n)$, and it was shown that it is sufficient and necessary for each node to be connected to its $\Theta(\log n)$ nearest neighbors in order to achieve asymptotic connectivity for this network. An immediate thought was whether one could simply replace n with $np(n)$ in this result to obtain the conditions for a network of the type $\mathcal{G}_p(n, R(n))$, assuming $np(n) \to \infty$. Although intuitively appealing, there is a conceptual difference. Replacing n with $np(n)$ in this result implies that the sufficient and necessary conditions for asymptotic connectivity are for every active node to be connected to $np(n)$ nearest active neighbors. However, these conditions are not directly guaranteed when the neighborhood of each node is defined by a fixed radius $R(n)$ with randomly deployed nodes, and when the nodes are randomly duty-cycled. Instead, what Theorem 1 shows is that it is sufficient and necessary for each active node to be connected to an average of $\Theta(\log(np(n)))$ active neighbors for asymptotic connectivity of a network of the type $\mathcal{G}_p(n, R(n))$.

In [3] a network of the type $\mathcal{G}(n, R(n))$ was considered, and it was shown that with $\pi R^2(n) = \frac{\log n + c(n)}{n}$, the network is asymptotically connected with probability one if and only if $c(n) \to \infty$. This result is not directly used in our study. However, throughout this paper we follow heavily the basic definitions and methods used by [1] and [3], as well as use a number of (intermediate) results derived in them with appropriate modifications. These will be pointed out in subsequent sections.

[4] showed that the sufficient and necessary conditions for asymptotic coverage with connectivity in a *grid* network are $p(n)R^2(n) = \Theta(\frac{\log n}{n})$. Although mathematically similar, these conditions are not the same as the ones given by Theorem 1, since asymptotic coverage with connectivity is a different measure from asymptotic connectivity, and a grid network is different from a random network. [4] also showed that the sufficient condition for asymptotic connectivity in the grid network is in the form of

$$np(n)e^{-\frac{\pi p(n)R^2(n)n}{2}} \to 0 \text{ as } n \to \infty.$$

It can be shown that $p(n)R^2(n) = \Theta(\frac{\log n}{n})$ implies $np(n)e^{-\frac{\pi p(n)R^2(n)n}{2}} \to 0$ as n and $np(n)$ both go to infinity. The reverse is not necessarily true. Therefore, we see that the condition for a randomly deployed network, i.e., $p(n)R^2(n) = \Theta(\frac{\log n}{n})$, is more restrictive than that for a grid network. Other related work includes [5], which studied the necessary and sufficient conditions of both asymptotic coverage and connectivity for a network with fixed node density λ but increasing area A. In addition, the concepts

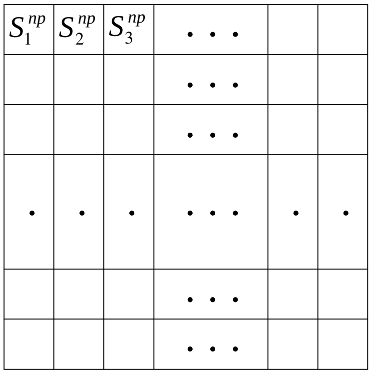

Fig. 2. The square tessellation τ_S^{np}.

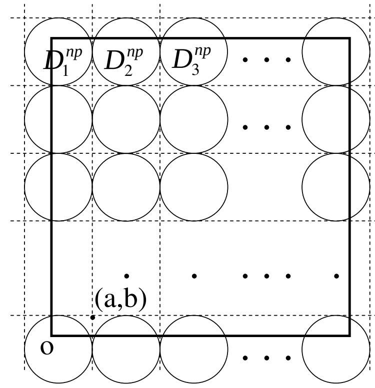

Fig. 3. The disk tessellation τ_D^{np}.

of k-connectivity and path connectivity were studied in [6] and [7], respectively.

III. PRELIMINARIES

For the proof of Theorem 1, we need the following definitions which were originally defined in [1], with slight generalization to account for $p(n) < 1$.

Definition 1 *Square tessellation* τ_S^{np}. The unit square is split equally into $M_{np} = \lceil \sqrt{\frac{np(n)}{K \log(np(n))}} \rceil^2$ small squares as depicted in Figure 2, where a constant $K > 0$ is a tunable parameter, and $\lceil x \rceil$ is the smallest integer larger than or equal to x. This tessellation of the unit square will be denoted by τ_S^{np}. The small squares are denoted by $S_i^{np}, i = 1, 2, \cdots, M_{np}$, from left to right, and from top to bottom.

Definition 2 *k-filling event.* Consider a structure composed of 21 squares each of side length $d/6$ and placed in a larger square of side length d: one at the center and the others at the periphery of the larger square with distance $d/4$ between the center square and the others. A k-filling event occurs if there are at least k nodes in each of 21 small squares and no nodes in the space between the center square and the others.

Definition 3 *Disk tessellation* $\tau_D^{np}(a, b)$. Consider a unit square with its bottom left corner being the origin, as shown in Figure 3. Let r be such that $\pi r^2 = \frac{K \log(np(n))}{np(n)}$, where $K > 0$ is a tunable parameter. Consider a grid of squares of size $2r$, with corners at ($a \mod 2r$, $b \mod 2r$). Inside each square, we inscribe a disk of area $\frac{K \log(np(n))}{np(n)}$. The set of all disks intersecting the unit square are called the Disk Tessellation $\tau_D^{np}(a, b)$. The disks intersecting the unit square are denoted by $D_i^{np}, i = 1 \leq M_{np}$.

Throughout our analysis, the asymptotic regime of interest is where the duty cycle $p(n) \to 0$, $n \to \infty$ and $np(n) \to \infty$.

Consider the network $\mathcal{G}^\lambda(np(n), R(n))$, where $0 < p(n) < 1$. Denote the number of nodes that fall into the unit square by $\widetilde{M}_{np}$, and denote the number of nodes that fall into square S_i^{np} by $\widetilde{N}_i^{np}$.

Lemma 1 $\lim_{np(n) \to \infty} Pr\{|\widetilde{M}_{np} - np(n)| \leq \sqrt{np(n) \log(np(n))}\} = 1$.

Consider $\mathcal{G}_p(n, R(n))$. Denote the number of active nodes in the unit square by M_n^a, which is a random variable. Denote the number of active nodes in square S_i^{np} by N_i^a.

Lemma 2 $\lim_{np(n) \to \infty} Pr\{|M_n^a - np(n)| \leq \sqrt{np(n) \log(np(n))}\} = 1$.

Lemma 3 *Suppose that* $p(n) \to 0$ *and* $np(n) \to \infty$ *as* $n \to \infty$. *For any nonnegative* $j \leq n$ *and sufficiently large* n, $Pr\{M_n^a = j\}$ *is approximated by* $Pr\{\widetilde{M}_{np} = j\}$, *i.e., in the limit their difference goes to zero.*

Proof: We have that

$$Pr\{M_n^a = j\} = \binom{n}{j} p(n)^j (1 - p(n))^{n-j} ,$$

$$Pr\{\widetilde{M}_{np} = j\} = \frac{(np(n))^j e^{-np(n)}}{j!}.$$

As $Pr\{M_n^a = j\}$ is a binomial distribution determined by n and $p(n)$, we will denote it by $b(j; n, p(n))$. Thus

$$b(0; n, p(n)) = (1 - p(n))^n. \tag{3}$$

By the definition of the derivative of function $\log x$, we have

$$\lim_{\delta \to 0} \frac{\log x - \log(x - \delta)}{\delta} = \frac{1}{x} . \tag{4}$$

Since $p(n) \to 0$ as $n \to \infty$, Eqn. (4) can be written as

$$\lim_{n\to\infty} \frac{\log x - \log(x - p(n))}{p(n)} = \frac{1}{x}.$$

For $x = 1$, we have

$$\lim_{n\to\infty} \frac{-\log(1 - p(n))}{p(n)} = 1.$$

In other words, $\forall \epsilon_1 > 0$, there exists $N_1 > 0$ such that $n > N_1$ implies $|\frac{-\log(1-p(n))}{p(n)} - 1| < \epsilon_1$. Let $\Delta(n) \equiv \frac{-\log(1-p(n))}{p(n)} - 1$, such that $\Delta(n) \in [-\epsilon_1, \epsilon_1]$. For all $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exists $N_2 > 0$ such that $n > \max\{N_1, N_2\}$ implies

$$\begin{aligned}
& |(1 - p(n))^n - e^{-np(n)}| \\
= \;& |(1 - p(n))^{-\frac{1}{p(n)} \cdot (-np(n))} - e^{-np(n)}| \\
= \;& |e^{\frac{-1}{p(n)} \log(1-p(n)) \cdot (-np(n))} - e^{-np(n)}| \\
= \;& |e^{(1+\Delta(n)) \cdot (-np(n))} - e^{-np(n)}| \\
= \;& |e^{-np(n)}(e^{-np(n) \cdot \Delta(n)} - 1)|. \qquad (5)
\end{aligned}$$

Because $|\Delta(n)|$ is bounded by ϵ_1, $|e^{-np(n)\cdot\Delta(n)} - 1|$ is bounded by some $N_3 > 0$. Therefore, Eqn. (5) $\leq |e^{-np(n)}| \cdot N_3 < \epsilon_2$. Thus for sufficiently large n we have

$$b(0; n, p(n)) \approx e^{-np(n)}.$$

Furthermore, for any fixed j we have

$$\frac{b(j; n, p(n))}{b(j - 1; n, p(n))} = \frac{np(n) - (j - 1)p(n)}{j(1 - p(n))}.$$

Therefore for sufficiently large n, we have

$$b(j; n, p(n)) \approx \frac{(np(n))^j}{j!} e^{-np(n)} = Pr\{\widetilde{M}_{np} = j\}.$$

■

Lemma 4 *For any* $K > \frac{1}{log(4/e)}$,
$\lim_{np(n)\to\infty} Pr\{\max_i |\widetilde{N}_i^{np} - K\log(np(n))| \leq \mu K \log(np(n))\} = 1, \forall \mu \in (\mu^*, 1)$,
where $\mu^* \in (0, 1)$ *is the sole root of the equation* $-\mu^* + (1 + \mu^*)\log(1 + \mu^*) = \frac{1}{K}$.

IV. PROOF OF THEOREM 1

In this section, we prove both two parts of Theorem 1. For simplicity we will ignore edge effect in our discussion, but note that edge effect does not alter the main theorem (see also [1,3]). The proof of each part consists of three steps. In part I, the proof proceeds as follows:

(1) Given $np(n)R(n)^2 = k_2\log(np(n))$ for some $k_2 > 0$, we show $\mathcal{G}_p(n, R(n))$ is asymptotically connected if $\mathcal{G}^\lambda(np(n), R(n))$ is asymptotically connected.

(2) It is shown that if there exists c_2 such that $\mathcal{F}^\lambda(np(n), c_2\log(np(n)))$ is asymptotically connected, then there exists k_2 such that $\mathcal{G}^\lambda(np(n), R(n))$ is asymptotically connected with $np(n)R(n)^2 = k_2\log(np(n))$.

(3) We show that $\mathcal{F}^\lambda(np(n), c_2\log(np(n)))$ is asymptotically connected for some $c_2 > 0$.

For the first step, note that $R(n)$ is bounded and that $n \to \infty$ implies $np(n) \to \infty$. For sufficiently large n,

$$\begin{aligned}
& Pr\{\mathcal{G}_p(n, R(n)) \text{ is connected}\} \\
&= \sum_{j=0}^{n} Pr\{\mathcal{G}_p(n, R(n)) \text{ is connected} | M_n^a = j\} \\
&\quad \cdot Pr\{M_n^a = j\} \\
&= \Big(\sum_{|j-np(n)| \leq \sqrt{(np(n)\log(np(n)))}} + \sum_{otherwise} \Big) \\
&\quad Pr\{\mathcal{G}_p(n, R(n)) \text{ is connected} | M_n^a = j\} \\
&\quad \cdot Pr\{M_n^a = j\} \\
&= \sum_{|j-np(n)| \leq \sqrt{(np(n)\log(np(n)))}} Pr\{\mathcal{G}_p(n, R(n)) \text{ is} \\
&\quad \text{connected} | M_n^a = j\} \cdot Pr\{M_n^a = j\} + o(1) \\
&= \sum_{|j-np(n)| \leq \sqrt{(np(n)\log(np(n)))}} Pr\{\mathcal{G}^\lambda(np(n), R(n)) \\
&\quad \text{is connected} | \widetilde{M}_{np} = j\} \cdot Pr\{\widetilde{M}_{np} = j\} \\
&\quad \cdot (1 + o(1)) + o(1)\ , \qquad (6)
\end{aligned}$$

where the third equality is based on Lemma 1. The fourth equality is based on Lemma 3 and the fact that $\mathcal{G}_p(n, R(n))$ given j active nodes is the same as $\mathcal{G}^\lambda(np(n), R(n))$ given j nodes are in the network. From Lemma 2 we have that Eqn. (6) can be written as

$$\begin{aligned}
& (1 + o(1)) \cdot (Pr\{\mathcal{G}^\lambda(np(n), R(n)) \text{ is connected}\} \\
& \quad + o(1)) + o(1).
\end{aligned}$$

Therefore if

$$\lim_{n\to\infty} Pr\{\mathcal{G}^\lambda(np(n), R(n)) \text{ is connected}\} = 1,$$

then

$$\lim_{n\to\infty} Pr\{\mathcal{G}_p(n, R(n)) \text{ is connected}\} = 1\ ,$$

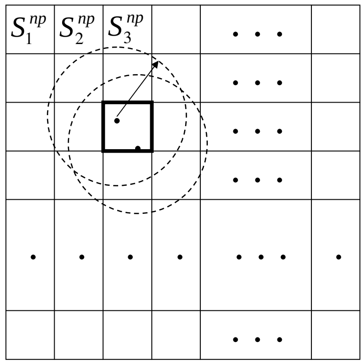

Fig. 4. Nodes with radius of transmission $R(n) = \sqrt{\frac{2K\log(np(n))}{np(n)}}$ on τ_S^{np}.

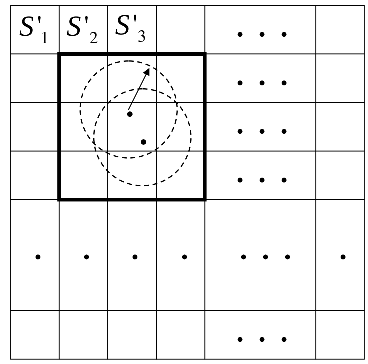

Fig. 5. Nodes with radius of transmission $R(n) = \sqrt{\frac{K'\log(np(n))}{np(n)}}$ on $\tau_{S'}^{np}$.

thus completing the first step.

In Step 2 we show that if there exists c_2 for $\mathcal{F}^\lambda(np(n), c_2\log(np(n)))$ to be asymptotically connected, then there exists k_2 for $\mathcal{G}^\lambda(np(n), R(n))$ to be asymptotically connected with $np(n)R(n)^2 = k_2\log(np(n))$. To prove this, let us tessellate $\mathcal{G}^\lambda(np(n), R(n))$ by τ_S^{np}, with K, μ satisfying Lemma 4. Consider some nodes whose radius is $R(n) = \sqrt{\frac{2K\log(np(n))}{np(n)}}$ on τ_S^{np}, as shown in Figure 4. Every circle contains a small square. From Lemma 4, we know that each circle contains more than or equal to $K(1-\mu)\log(np(n))$ nodes with high probability, where $\mu \in (\mu^*, 1)$. We construct another graph by connecting each node with its nearest $K(1-\mu)\log(np(n)) - 1$ neighbors, which is $\mathcal{F}^\lambda(np(n), K(1-\mu)\log(np(n)) - 1)$. If $\mathcal{F}^\lambda(np(n), K(1-\mu)\log(np(n)) - 1)$ is asymptotically connected, then $\mathcal{G}^\lambda(np(n), R(n))$ with $np(n)R(n)^2 = 2K\log(np(n))$ is asymptotically connected. Thus there exists $k_2 = 2K$ when $c_2 = K(1-\mu)$. This completes the second step.

Finally, we want to prove that $\mathcal{F}^\lambda(np(n), c_2\log(np(n)))$ is asymptotically connected for $c_2 > \frac{2}{\log(4/e)}$. It suffices to show that for some $\delta > 0$,

$$\lim_{n\to\infty} Pr\{\mathcal{F}^\lambda(np(n), (2/\log(4/e) + \delta)\log(np(n))) \text{ is connected}\} = 1.$$

This proof is similar to that in [1] and is thus not detailed here.

The proof of the second part of Theorem 1 follows a very similar procedure, consisting of three steps:

(1) Given $np(n)R(n)^2 = k_1\log(np(n))$ for some $k_1 > 0$, we show $\mathcal{G}_p(n, R(n))$ is asymptotically disconnected if $\mathcal{G}^\lambda(np(n), R(n))$ is asymptotically disconnected.

(2) It is shown that if there exists c_1 such that $\mathcal{F}^\lambda(np(n), c_1\log(np(n)))$ is asymptotically disconnected, then there exists k_1 such that $\mathcal{G}^\lambda(np(n), R(n))$ is asymptotically disconnected with $np(n)R(n)^2 = k_1\log(np(n))$.

(3) We show that $\mathcal{F}^\lambda(np(n), c_1\log(np(n)))$ is asymptotically disconnected for some $c_1 > 0$.

In the first step, similar to part I we will use the fact that $n \to \infty$ implies $np(n) \to \infty$. With slight modification from connectivity to disconnectivity on the argument used in part I of the proof given early, one can easily show that if $\lim_{n\to\infty} Pr\{\mathcal{G}^\lambda(np(n), R(n)) \text{ is disconnected}\} = 1$, then $\lim_{n\to\infty} Pr\{\mathcal{G}_p(n, R(n)) \text{ is disconnected}\} = 1$. This completes the first step of the proof of part II.

In the second step we show that if there exists c_1 such that $\mathcal{F}^\lambda(np(n), c_1\log(np(n)))$ is asymptotically disconnected, then there exists k_1 such that $\mathcal{G}^\lambda(np(n), R(n))$ with $np(n)R(n)^2 = k_1\log(np(n))$ is asymptotically disconnected. To prove this, we tessellate $\mathcal{G}^\lambda(np(n), R(n))$ by τ_S^{np}, with K, μ satisfying Lemma 4. Furthermore, we split each square into $\lceil\sqrt{\frac{9\cdot 21(1+\mu)}{1-\mu}}\rceil^2$ smaller squares. Denote by $\tau_{S'}^{np}$ the new tessellation with $\lceil\sqrt{\frac{np(n)}{K\log(np(n))}}\rceil^2 \cdot \lceil\sqrt{\frac{9\cdot 21(1+\mu)}{1-\mu}}\rceil^2$ squares and let $\widetilde{N}_i^*$ be the number of nodes in each smaller square S_i'. Thus $\widetilde{N}_i^*$ is a Poisson random variable with mean $\frac{K(1-\mu)}{9\cdot 21(1+\mu)}\log(np(n))$. Similarly to Lemma 4, for $K' > \frac{1}{\log(4/e)}$, we have

$$\lim_{n\to\infty} Pr\{\max_i \widetilde{N}_i^* \le (1+\mu)K'\log(np(n))\} = 1, \quad \forall \mu \in (\mu^{**}, 1), \tag{7}$$

where $K' = \frac{1-\mu}{9\cdot 21(1+\mu)}K$ and μ^{**} is the root of $-\mu^{**} + (1 + \mu^{**})\log(1+\mu^{**}) = \frac{1}{K'}$.

Consider some nodes with radius $R(n) = \sqrt{\frac{K' \log(np(n))}{np(n)}}$, the side length of each smaller square on $\tau_{S'}^{np}$ as shown in Figure 5. Every circle is included in a group of at most 9 small squares. From Eqn. (7), each circle contains less than or equal to $\frac{K(1-\mu)}{21} \log(np(n))$ nodes with high probability. We can thus construct another graph by connecting each node with its nearest $\frac{K(1-\mu)}{21} \log(np(n)) - 1$ neighbors, which results in $\mathcal{F}^{\lambda}(np(n), \frac{K(1-\mu)}{21} \log(np(n)) - 1)$. Consequently, if $\mathcal{F}^{\lambda}(np(n), \frac{K(1-\mu)}{21} \log(np(n)) - 1)$ is asymptotically disconnected, $\mathcal{G}^{\lambda}(np(n), R(n))$ with $np(n)R(n)^2 = \frac{1-\mu}{9 \cdot 21(1+\mu)} K \log(np(n))$ is asymptotically disconnected. Note that for large $np(n)$, $\frac{K(1-\mu)}{21} \log(np(n)) \gg 1$. Thus there exists $k_1 = \frac{1-\mu}{9 \cdot 21(1+\mu)} K$ when $c_2 = \frac{K(1-\mu)}{21}$. This completes the second step of the proof.

Finally, we want to prove that $\mathcal{F}^{\lambda}(np(n), c_1 \log(np(n)))$ is asymptotically disconnected for $c_1 < \frac{(1-\mu)K}{21}$. It suffices to show that for some $\epsilon > 0$,

$$\lim_{n \to \infty} Pr\{\mathcal{F}^{\lambda}(np(n), \epsilon \log(np(n)))) \text{ is connected}\} = 0.$$

Again this proof is similar to that in [1] and is not detailed here.

V. CONCLUSION

In this paper we studied the asymptotic connectivity of a low duty-cycled wireless sensor network where sensor nodes are randomly duty-cycled according to a fixed active probability. We derived the sufficient and necessary conditions for the network to be connected as the number of node grows to infinity. These conditions are in the form of the joint scaling behavior of the number of nodes in the network as well as the active probability. Thus such results reveal how duty-cycling should be scaled as the network gets denser in order to maintain network connectivity.[1]

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